> **Remark** (Why the gamma function?)**.** The gamma distribution is the *parent* of both the exponential distribution and the chi-squared distribution, and it is closely tied to the Poisson process. Most of the work is pure-maths revision – integration by parts and reduction arguments.

## Extending the Factorial

The factorial $n!$ is defined for non-negative integers. Is there a natural function of a *real* variable that passes through the factorials? Euler found one, written as an integral:

**Definition.** The **gamma function** is defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\,\mathrm{d}t$$

> **Remark** (Why $x > 0$?)**.** Near $t = \infty$ the factor $e^{-t}$ crushes any power of $t$, so the tail always converges. Near $t = 0$ the integrand behaves like $t^{x-1}$, and $\int_0^1 t^{x-1}\,\mathrm{d}t$ converges if and only if $x > 0$. (With complex analysis the definition can be extended to all complex numbers except $0, -1, -2, \ldots$ – see the closing remark.)

**Theorem** (The functional equation)

For all $x > 0$,

$$\Gamma(x+1) = x\Gamma(x)$$

**Example**

Prove this, using integration by parts.

*Integrate by parts with $u = t^x$, $\mathrm{d}v = e^{-t}\,\mathrm{d}t$:*

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t}\,\mathrm{d}t = \left[-t^x e^{-t}\right]_0^\infty + \int_0^\infty x t^{x-1} e^{-t}\,\mathrm{d}t$$

*The bracket vanishes: at $t = 0$ because $x > 0$, and as $t \to \infty$ because exponentials beat powers. Hence*

$$\Gamma(x+1) = x\int_0^\infty t^{x-1} e^{-t}\,\mathrm{d}t = x\Gamma(x) \qquad \blacksquare$$

*This is precisely the recurrence $n! = n \times (n-1)!$ in disguise.*

**Example**

Show that $\Gamma(1) = 1$, and deduce that $\Gamma(n) = (n-1)!$ for every positive integer $n$.

$$\Gamma(1) = \int_0^\infty t^0 e^{-t}\,\mathrm{d}t = \left[-e^{-t}\right]_0^\infty = 1$$

*Then by the functional equation, repeatedly:*

$$\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)(n-2)\Gamma(n-2) = \cdots = (n-1)(n-2)\cdots 1\cdot\Gamma(1) = (n-1)!$$

*(Formally, induction on $n$.) Note the irritating shift: $\Gamma(n) = (n-1)!$, not $n!$ – a historical accident we are stuck with.*

**Example** (Class practice)

Evaluate $\int_0^\infty x^5 e^{-x}\,\mathrm{d}x$ and $\int_0^\infty x^3 e^{-2x}\,\mathrm{d}x$.

*The first is $\Gamma(6) = 5! = 120$ on sight. For the second, substitute $u = 2x$, $\mathrm{d}u = 2\,\mathrm{d}x$:*

$$\int_0^\infty \left(\frac{u}{2}\right)^3 e^{-u}\,\frac{\mathrm{d}u}{2} = \frac{1}{2^4}\int_0^\infty u^3 e^{-u}\,\mathrm{d}u = \frac{\Gamma(4)}{16} = \frac{3!}{16} = \frac{3}{8}$$

*This substitution trick is about to become a lemma.*

## Half-Integer Values

What is $\Gamma\left(\frac{1}{2}\right)$ – morally, "$\left(-\frac{1}{2}\right)!$"? The answer is one of the most famous in mathematics.

**Fact** (The Gaussian integral) —

$$\int_{-\infty}^{\infty} e^{-u^2}\,\mathrm{d}u = \sqrt{\pi}$$

(Sketch of the standard proof: call the integral $I$; then $I^2 = \iint e^{-(u^2+v^2)}\,\mathrm{d}u\,\mathrm{d}v$ over the whole plane, which in polar coordinates becomes $\int_0^{2\pi}\int_0^{\infty} e^{-r^2}\,r\,\mathrm{d}r\,\mathrm{d}\theta = 2\pi \times \frac{1}{2} = \pi$. This is also the fact that makes the normal pdf integrate to 1.)

**Theorem**

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

**Example**

Prove this, using the substitution $t = u^2$.

*With $t = u^2$ (for $u > 0$), $\mathrm{d}t = 2u\,\mathrm{d}u$ and $t^{-1/2} = 1/u$:*

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{\infty} t^{-1/2} e^{-t}\,\mathrm{d}t = \int_0^{\infty} \frac{1}{u} e^{-u^2} \cdot 2u\,\mathrm{d}u = 2\int_0^{\infty} e^{-u^2}\,\mathrm{d}u = \int_{-\infty}^{\infty} e^{-u^2}\,\mathrm{d}u = \sqrt{\pi}$$

*using the symmetry of $e^{-u^2}$ in the last step.* ■

**Example**

Evaluate $\Gamma\left(\frac{7}{2}\right)$.

*Apply the functional equation repeatedly:*

$$\Gamma\left(\tfrac{7}{2}\right) = \tfrac{5}{2}\Gamma\left(\tfrac{5}{2}\right) = \tfrac{5}{2}\cdot\tfrac{3}{2}\Gamma\left(\tfrac{3}{2}\right) = \tfrac{5}{2}\cdot\tfrac{3}{2}\cdot\tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \frac{15\sqrt{\pi}}{8}$$

**Example** (Class practice)

Evaluate $\Gamma\left(\frac{9}{2}\right)$ and $\displaystyle\int_0^{\infty} \sqrt{x}\,e^{-x}\,\mathrm{d}x$.

*$\Gamma\left(\frac{9}{2}\right) = \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi} = \dfrac{105\sqrt{\pi}}{16}$, and $\displaystyle\int_0^\infty x^{1/2}e^{-x}\,\mathrm{d}x = \Gamma\left(\tfrac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}$.*

The integrand $x^n e^{-x^2}$ below should look familiar: it is exactly what the substitution $t = u^2$ produced in the proof of $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, only here over a finite interval.

**Example** (CAIE FP1, November 2002)
It is given that, for $n \geq 0$,

$$I_n = \int_0^1 x^n e^{-x^2}\,\mathrm{d}x.$$

(i) Find $I_1$ in terms of $e$.

(ii) Show that

$$I_{n+2} = \frac{n+1}{2}I_n - \frac{1}{2e}.$$

(iii) Find $I_5$ in terms of $e$.

*(i) $xe^{-x^2}$ has antiderivative $-\frac{1}{2}e^{-x^2}$ on sight, so*

$$I_1 = \left[-\frac{1}{2}e^{-x^2}\right]_0^1 = \frac{1}{2} - \frac{1}{2e}.$$

*(ii) Peel off one factor of $x$ and integrate by parts with $u = x^{n+1}$, $\mathrm{d}v = xe^{-x^2}\,\mathrm{d}x$, $v = -\frac{1}{2}e^{-x^2}$:*

$$I_{n+2} = \int_0^1 x^{n+1}\cdot xe^{-x^2}\,\mathrm{d}x = \left[-\frac{1}{2}x^{n+1}e^{-x^2}\right]_0^1 + \frac{n+1}{2}\int_0^1 x^n e^{-x^2}\,\mathrm{d}x = -\frac{1}{2e} + \frac{n+1}{2}I_n.$$

*(iii) Apply the reduction formula twice:*

$$I_3 = \frac{2}{2}I_1 - \frac{1}{2e} = \frac{1}{2} - \frac{1}{e}, \qquad I_5 = \frac{4}{2}I_3 - \frac{1}{2e} = 1 - \frac{2}{e} - \frac{1}{2e} = 1 - \frac{5}{2e}.$$

*Compare part (ii) with the proof of the functional equation: same integration by parts, but the finite upper limit leaves the boundary term $-\frac{1}{2e}$ behind. Letting the upper limit tend to $\infty$ instead, the boundary term vanishes and the substitution $t = x^2$ turns $\int_0^\infty x^n e^{-x^2}\,\mathrm{d}x$ into $\frac{1}{2}\Gamma\left(\frac{n+1}{2}\right)$.*

## The Gamma Distribution

The gamma function lets us build a whole family of pdfs. First, generalise the substitution trick from earlier:

**Lemma**

For $\alpha, \beta > 0$,

$$\int_0^\infty x^{\alpha-1} e^{-\beta x}\,\mathrm{d}x = \frac{\Gamma(\alpha)}{\beta^\alpha}$$

The proof is a one-line substitution.

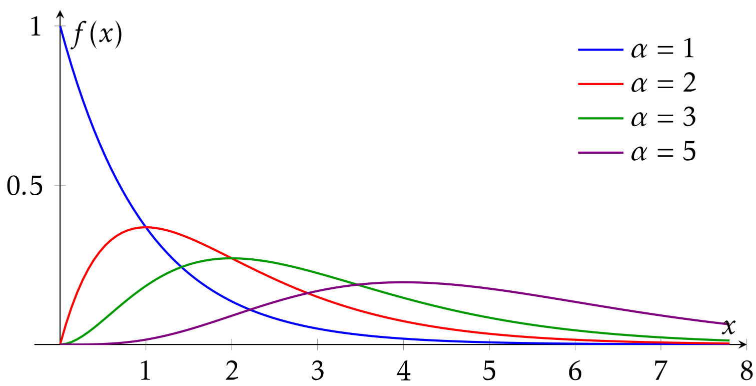
*Substitute $u = \beta x$, so $x = u/\beta$ and $\mathrm{d}x = \mathrm{d}u/\beta$:*

$$\int_0^\infty \left(\frac{u}{\beta}\right)^{\alpha-1} e^{-u}\,\frac{\mathrm{d}u}{\beta} = \frac{1}{\beta^\alpha}\int_0^\infty u^{\alpha-1}e^{-u}\,\mathrm{d}u = \frac{\Gamma(\alpha)}{\beta^\alpha} \qquad \blacksquare$$

Dividing through by $\Gamma(\alpha)/\beta^\alpha$ gives a non-negative function with total integral 1 – that is, a pdf:

**Definition.** The random variable $X$ has the **gamma distribution** with *shape* $\alpha > 0$ and *rate* $\beta > 0$, written $X \sim \Gamma(\alpha, \beta)$, if

$$f(x) = \begin{cases} \dfrac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$




Gamma pdfs with rate $\beta = 1$: for $\alpha = 1$ we get exponential decay; for larger $\alpha$ a skewed hump that drifts right and becomes more symmetric.

**Theorem**

If $X \sim \Gamma(\alpha, \beta)$ then

$$\mathbb{E}[X] = \frac{\alpha}{\beta} \qquad \text{and} \qquad \mathrm{Var}[X] = \frac{\alpha}{\beta^2}$$

**Example**

Prove this. (No integration by parts needed – use the lemma and the functional equation.)

$$\mathbb{E}[X] = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^\alpha e^{-\beta x}\,\mathrm{d}x = \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+1)}{\beta^{\alpha+1}} = \frac{\alpha\,\Gamma(\alpha)}{\beta\,\Gamma(\alpha)} = \frac{\alpha}{\beta}$$

$$\mathbb{E}[X^2] = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha+1} e^{-\beta x}\,\mathrm{d}x = \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+2)}{\beta^{\alpha+2}} = \frac{(\alpha+1)\alpha}{\beta^2}$$

*using* $\Gamma(\alpha+2) = (\alpha+1)\Gamma(\alpha+1) = (\alpha+1)\,\alpha\,\Gamma(\alpha)$. *Hence*

$$Var[X] = \frac{\alpha^2+\alpha}{\beta^2} - \frac{\alpha^2}{\beta^2} = \frac{\alpha}{\beta^2} \qquad \blacksquare$$

## Special Cases and the Poisson Process

**Fact —**

- **Exponential:** $\Gamma(1,\lambda)$ has pdf $\frac{\lambda^1}{\Gamma(1)}x^0 e^{-\lambda x} = \lambda e^{-\lambda x}$, so $\text{Exp}(\lambda) = \Gamma(1,\lambda)$. (Check: mean $\frac{1}{\lambda}$, variance $\frac{1}{\lambda^2}$. ✓)
- **Chi-squared:** the chi-squared distribution with $k$ degrees of freedom (next chapter) is $\chi^2_k = \Gamma\left(\frac{k}{2},\frac{1}{2}\right)$ – which is why its pdf involves $\Gamma\left(\frac{k}{2}\right)$, and why half-integer values of $\Gamma$ matter. Its mean is $\frac{k/2}{1/2} = k$ and variance $\frac{k/2}{1/4} = 2k$.

**Theorem** (Waiting time for the $n$th arrival)
In a Poisson process with rate $\lambda$, the waiting time $T_n$ until the $n$th occurrence has distribution $\Gamma(n,\lambda)$.

**Example**
Prove this for general $n$, by finding $\mathbb{P}(T_n > t)$ and differentiating. (We did the case $n = 1$ – the exponential distribution – in the continuous random variables chapter.)

*The nth arrival happens after time t exactly when at most $n-1$ arrivals occur in $[0,t]$, and the number of arrivals in $[0,t]$ is Po($\lambda t$):*

$$\mathbb{P}(T_n > t) = \sum_{k=0}^{n-1} \frac{e^{-\lambda t}(\lambda t)^k}{k!}$$

*So $F(t) = 1 - \sum_{k=0}^{n-1} \frac{e^{-\lambda t}(\lambda t)^k}{k!}$, and differentiating term by term (product rule on each term):*

$$f(t) = -\sum_{k=0}^{n-1}\left(\frac{-\lambda e^{-\lambda t}(\lambda t)^k}{k!} + \frac{e^{-\lambda t}\,k\lambda(\lambda t)^{k-1}}{k!}\right) = \sum_{k=0}^{n-1} \lambda e^{-\lambda t}\left(\frac{(\lambda t)^k}{k!} - \frac{(\lambda t)^{k-1}}{(k-1)!}\right)$$

*The sum telescopes (the $k = 0$ term has no second part), leaving only the top term:*

$$f(t) = \lambda e^{-\lambda t}\frac{(\lambda t)^{n-1}}{(n-1)!} = \frac{\lambda^n}{\Gamma(n)} t^{n-1} e^{-\lambda t}$$

*which is exactly the $\Gamma(n,\lambda)$ pdf.* ■

*Consistency check: a sum of n independent Exp($\lambda$) gaps should have mean $\frac{n}{\lambda}$ and variance $\frac{n}{\lambda^2}$ – and indeed $\Gamma(n,\lambda)$ has mean $\frac{n}{\lambda}$ and variance $\frac{n}{\lambda^2}$.*

**Example**
A radioactive source emits particles as a Poisson process at a rate of 2 per second. Find the probability that the third particle is emitted within the first 2 seconds.

*$T_3 \sim \Gamma(3,2)$, and by the waiting-time argument*

$$\mathbb{P}(T_3 \le 2) = 1 - \mathbb{P}(\textit{at most 2 emissions in } [0,2]) = 1 - e^{-4}\left(1 + 4 + \frac{4^2}{2}\right) = 1 - 13e^{-4} \approx 0.762$$

*(Far easier than integrating the gamma pdf – though integrating $\int_0^2 \frac{2^3}{\Gamma(3)} t^2 e^{-2t}\,\mathrm{d}t$ by parts twice gives the same answer.)*

## The MGF of the gamma distribution

If you studied the moment generating functions chapter, the gamma distribution ties everything together.

> **Example**
>
> Show that if $X \sim \Gamma(\alpha,\beta)$ then $M_X(t) = \left(\dfrac{\beta}{\beta - t}\right)^{\alpha}$ for $t < \beta$. Deduce that the sum of independent gamma variables with the *same rate* is gamma: $\Gamma(\alpha_1,\beta) + \Gamma(\alpha_2,\beta) \sim \Gamma(\alpha_1+\alpha_2,\beta)$.

*Using the lemma with rate $\beta - t$ (valid when $\beta - t > 0$):*

$$M_X(t) = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-(\beta-t)x}\,\mathrm{d}x = \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha)}{(\beta-t)^\alpha} = \left(\frac{\beta}{\beta-t}\right)^{\alpha}$$

*For independent $X \sim \Gamma(\alpha_1,\beta)$ and $Y \sim \Gamma(\alpha_2,\beta)$:*

$$M_{X+Y}(t) = \left(\frac{\beta}{\beta-t}\right)^{\alpha_1}\left(\frac{\beta}{\beta-t}\right)^{\alpha_2} = \left(\frac{\beta}{\beta-t}\right)^{\alpha_1+\alpha_2}$$

*which by uniqueness of MGFs is $\Gamma(\alpha_1+\alpha_2,\beta)$.* ■

*In particular, a sum of $n$ independent Exp($\lambda$) = $\Gamma(1,\lambda)$ variables is $\Gamma(n,\lambda)$ – a second, slicker proof of the waiting-time theorem, since $T_n$ is the sum of the $n$ exponential gaps between arrivals. The MGF also gives the mean and variance instantly: $M'(0) = \frac{\alpha}{\beta}$, $M''(0) - M'(0)^2 = \frac{\alpha}{\beta^2}$.*

> **Remark** (Shape–rate versus shape–scale)**.** We have used the *shape–rate* parametrisation $\Gamma(\alpha,\beta)$. Many texts (and most statistical software) instead use *shape–scale* parameters $(\alpha,\theta)$ where $\theta = 1/\beta$, giving pdf $\frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-x/\theta}$, mean $\alpha\theta$ and variance $\alpha\theta^2$. Always check which convention a source is using before quoting formulae.

> **Remark** (Further reading)**.** The gamma function is everywhere in higher mathematics. Extended to the complex plane it is intimately connected to the **Riemann zeta function** via
>
> $$\Gamma(s)\zeta(s) = \int_0^\infty \frac{t^{s-1}}{e^t - 1}\,\mathrm{d}t$$
>
> and it appears in the functional equation of $\zeta$, at the heart of the Riemann Hypothesis – the most famous unsolved problem in mathematics. See [Toller] Ch 9 and any introduction to analytic number theory.

**Textbook Exercises:** [Toller] Ch 9